The Fundamentals of Graph Theory

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Abstract

The field of Mathematics plays an important role to solve the real life and educational problems. It has many parts but graph theory is a main part of mathematics. A graph or network is a set of nodes and edges, where nodes are the individual elements within the network and edges represent connectivity between nodes. Edges may be binary (connected or not) or contain additional information about the level of connectivity. Networks surround us in both the natural and anthropogenic world. So we can say that graph theory is essential part of mathematics.

Keywords: Vertex, edge, bipartite, connectivity

Introduction

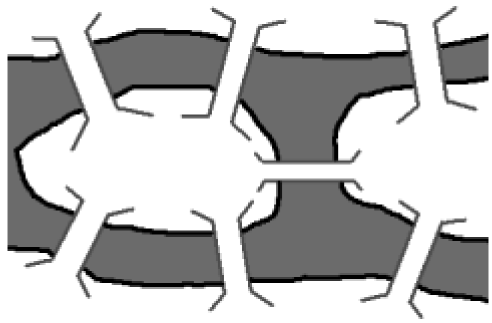
A graph is a collection of points (vertices) that are connected by lines (edges). Some graphs have direction, which means the lines have an arrow and only go in one direction. Some (rather boring) graphs can have no edges at all, in others the edges could overlap, and there can be many edges coming out of the same vertex.

Review of Literature

The origin of graph theory started with the problem of Konigsberg bridge, in 1735. The first textbook on graph theory was written by Dénes Kőnig[1], and published in 1936. This story is about the Swiss mathematician Leonhard Euler (1707 – 1783)[2]. While he lived in the Prussian town of Konigsberg at the Baltic sea, he was thinking about the following problem:

Konigsberg is divided into four parts by the river Pregel, and connected by seven bridges. Is it possible to tour Konigsberg along a path that crosses every bridge once, and at most once? You can start and finish wherever you want, not necessarily in the same place.

Euler was the first person to solve the famous problem "Konigsberg bridges problem", which tells you how to cross each of the seven bridges in the Figure below exactly once and return to your starting point.



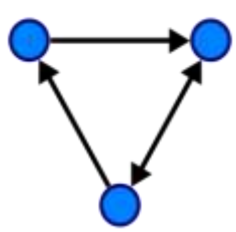
Types

1) Simple graph

A graph without loops and with at most one edge between any two vertices is called simple graph.

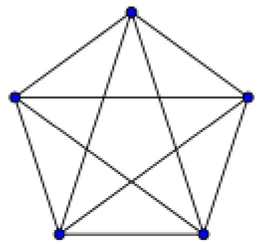
2) Directed Graphs and Mixed Graphs

Generally, the edges in a graph will not have directions associated with them. However in a directed graph, or digraph, each edge includes a direction from one endpoint to the other. In a mixed graph, both directed and undirected edges are allowed.



3) Complete Graphs

If all of the vertices in a graph are adjacent to each other, then the graph is called a complete graph. The symbol K_n is used to denote a complete graph with n vertices.

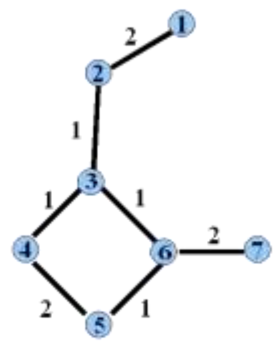


4) Connected and Disconnected Graphs

A graph is connected if every vertex is joined to every other vertex by a path. A disconnected graph is a graph that is not connected.

5) Weighted graph

A graph G{V,E} is called weighted graph if each edge of the graph G is assigned a number n>0 is called weight of the edge. Such weights might represent, for example, costs, lengths or capacities, etc. depending on the problem. Some authors call such a graph a network[3].



6) Multi-graph

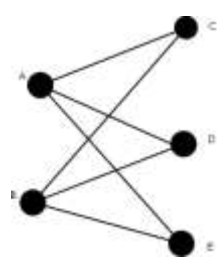
In a graph G{V,E} is called multi graph if there exit more than one edges between two vertices.

7) Bipartite graph

A graph G{V,E} is called bipartite graph if its vertices V can be partitioned into two subset X_1 and X_2 such that each edge of G connects a vertex of X_1 to a vertex of X_2. In other words , no edge joining two vertices in X_1 or two vertices in X_2. It is denoted by $K_{m,n}$ where m and n are number of vertices in X_1 and X_2 respectively.

or

A bipartite graph is a graph whose vertices can be separated into two sets X and Y in such a way that every edge in the graph has one endpoint in each set.



Representation of graph in matrix form

The essential information about a graph can be recorded in a matrix consisting of 0's and 1's in two different ways. The adjacency matrix of the graph indicates which vertices are adjacent to each other, while the incidence matrix indicates which edges occur at each vertex. Thus there are two way to represent the graph in matrix.

a) Undirected graph

The adjacency matrix

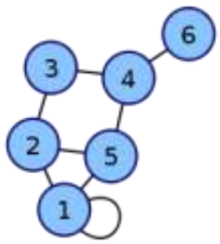
The adjacency matrix of the graph G{V,E} is an n × n matrix A = $[a_{ij}]$, where n is the number of vertices in G, V = $\{v_1,...,v_n\}$ and

$a_{ij=}$ n, where n is number of edges between v_i and v_j

and $a_{ij=}$ 0 if $\{v_i,v_j\}$ is not an edge in G.

Adjacency matrix of undirected graph is always is symmetric because if two vertices v_i and v_j has a edge then v_j and v_i has also a edge.

So $a_{ij} = a_{ji}$ for all i,j



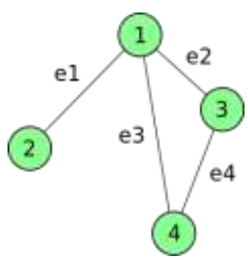
$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Incidence matrix

The all-vertex incidence matrix of a non-empty and loop less graph G{V,E} is an n×m matrix A = $[a_{ij}]$where n is the number of vertices in G, m is the number of edges in G and

$a_{ij=}$ 1 if v_i is an end vertex of e_j

and $a_{ij=}$ 0 otherwise



$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

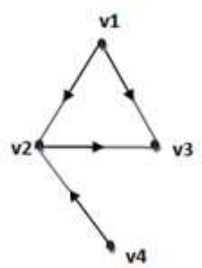
b) Directed graph

The adjacency matrix

The adjacency matrix of the graph G = (V,E) is an n × n matrix A = $[a_{ij}]$, where n is the number of vertices in G, V = $\{v_1,...,v_n\}$ and

$a_{ij} = 1$, if $\{v_i,v_j\}$ is an edge and v_i is initial vertex and v_j is final vertex

and $a_{ij} = 0$ if there is not an edge from v_i to v_j



$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

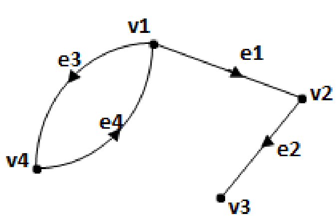
Incidence matrix

If a directed graph consist of n vertices and m edges then the incidence matrix of is an n × m matrix B = $[b_{ij}]$ is defined by

$b_{ij} = 1$, if v_i is initial vertex of edge e_j

and $b_{ij} = -1$, if v_i is final vertex of edge e_j

and $b_{ij} = 0$, if v_i is not an incident vertex on edge e_j



$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Applications of graph theory

In this world everything is connected to other i.e., cities are connected by road, rails and flight networking. We can use the graph theory in computer networking and traffic management. Scientists, engineers and many others want to analyze, understand and optimize to these networks. And this can be done using graph theory.

For example, mathematicians can apply graph theory to road networks, trying to find a way to reduce traffic congestion. An idea which, if successful, could save millions every year which are lost due to time spent on the road, as well as mitigating the enormous environmental impact. It could also make life safer by allowing emergency services to travel faster and avoid car accidents in the first place**.**

Applied areas

Topics	Areas
Tree	Chemistry, Electrical network
Directed graph	Communication theory, City construction planning
Planar graph	Electrical engineering, Map coloring

Graph theory can provide initial processing of landscape data and can serve as a guide to help develop and marshall landscapes scale plans, including the identification of sensitive areas across scales[4].

Graph theory plays a significant role in the modeling of biological systems. These models are used in the study of metabolic networks. Also, protein-protein interaction networks are basically represented in graphical format in which proteins are represented by vertices and their interactions by edges. Similarly, social structure and food webs have also been modeled using graph theory[5]. It is also capable of modeling the complex events with molecules, cells or living organisms constituent. Vertices in biological networks represent bio-molecules such as genes, proteins or metabolites and edges represent functional, physical or chemical interactions between the correspond bio-molecules. Thus, graph theoretic networks are also helpful in analyzing the interactions among these parts.

Graph theory provides a good framework in the study of molecules in chemistry. It can be applied to model a molecule in which atoms are represented by vertices and their bonds by edges[6]. Graph enumeration helps in describing a class of combinatorial enumeration problems in chemistry[7]. The credit goes to the famous mathematicians namely, Polya[8], Cayley[9] and Redfield[10]. Arthur Cayley (1821-1895) used the concept of a tree to describe the problem of hydrocarbon chemistry in finding isomers[11]. A main concept of graph theory known as " Cayley graph " have also been very useful for the researchers of chemistry in calculating the number of isomers. Similar approach of graph theory can also be used in quantum electronic structures, molecular mechanics and stimulated condensation phase, design of structure of molecules, potential energy, topography, polymers and biological macromolecules[12].

Graph theory is very useful in study in physical, biological, social and computer sciences. It has been used to solve many practical problems in these fields. With the help of graph theory, we can represent the network of communication and data organization in computer science. Directed graphs are used to represent the link structure of a website in which the vertex represents web pages and directed edges represent links from one page to another.

Graph theory also plays an important role in the evolution of animals and languages, crowd control and the spread of diseases.

Result

This theory has great importance in various fields of science like Computer science; Biological science; Social science; Chemical science etc. By the graphical representation in graph theory, gives us the mathematical data and make the work easy and actual. So this paper has great relevance for researchers who have keen interest in these fields.

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