VECTOR SPACE AND ITS SUBSPACE

Preeti

Assistant Professor of Mathematics

Govt. College Naraingarh, Ambala

Abstract:-

This article is about vector space and also known as linear vector space which is a collection of objects called vectors. They provide a framework to deal with analytical and geometrical problems, or are used in Fourier transform. In this Paper , we discuss vector space and it's subspaces. Vector space are the subject of linear algebra and are well characterized by their dimension (i.e specifies the number of independent direction in the space). Linear algebra is that branch of mathematics which treats the common properties of algebraic systems which consists of a set together with a reasonable notion of a 'linear combination ' of elements in the set. Historically, The first ideas leading to vector spaces can be traced back as far as the 17^{th} century's analytic geometry, matrices, system of linear equations and Euclidean vectors. Today, vector space are applied throughout mathematics, science and engineering and appropriate linear algebraic notion to deal with system of linear equations.

Keywords:-Vector space, Field, subspaces, Euclidean, Matrices.

Introduction :-

This paper introduces the concept of vector space. In reality, linear algebra is the study of vector spaces and the functions of vector spaces (linear transformation). They form the fundamental objects which we will studying throughout the remaining course. Once we define vector space, we will go on to study the properties of vector spaces. Their importance lies in the fact that many mathematical questions can be rephrasedas a question about vector space. Thus each fact that we prove about vector spaces gives us corresponding information about many different mathematical questions. The following notation will be used :

V the given vector space

u, v, w vectors in V

F the given number field

a, b, c, k scalars in F

A vector space is a set of vectors together with rules for vector addition and multiplication by real number. These operations must produce vectors in the space and must satisfy some conditions. Consider a set V endowed with two operation:

Addition:-given two vectors v1 and v2 in V, It associates a new vector in V,that will denote v1+v2; i.e +:V*V-------→V. (v1 , v2) ------→ v1+v2

Scalar multiplication:-given a vector v in V and a real number c, It associates a new vector in V, that we will denote c.v (or simply cv) i.e

*: R*V-------→V. (c, v) ----→ c.v.

In other words , A vector space that involves internal as well as external binary operations defined above .If we write elements of F then it is called scalars and if we write elements of V then it is called vectors.

<u>Definition of Vector space:-</u>

The following defines the notion of a vector space V where K is the field of scalars.

<u>DEFINITION</u>: Let V be a nonempty set with two operations:

(i) Vector Addition: This assigns to any u, v €V,(a sum u +v in V).

(ii) Scalar Multiplication: This assigns to any u €V, k € K a product ku €V.

Then V is called a vector space (over the field K) if the following axioms hold for any vectors u, v, w €V.

1. Closure under addition: For each pair of vectors u and v,the sum u + v is an element of V .

2. Closure under scalar multiplication: For each vector v andscalar k, the scalar multiple kv is an element of V .

3. Commutativity of addition: For all u, v ∈ V , we haveu + v = v + u.

4. Associativity of addition: For all u, v, w ∈ V , we have

(u + v) + w = u + (v + w).

5. Existence of a zero vector: There is a vector 0 ∈ Vsatisfying v + 0 = v for all v ∈ V

6. Existence of additive inverses: For each v ∈ V , there is a vector -v ∈ V such that v + (-v) = 0.

7. Unit property: For all vectors v, we have 1v = v.

8. Associativity of scalar multiplication: For all vectors v andscalars r, s, we have (rs)v = r(sv).

9. Distributive property of scalar multiplication over vectoraddition: For all vectors u and v and scalars r, we have r(u + v) = ru + rv.

10. Disributive property of scalar multiplication over scalaraddition: For all vectors v and scalars r and s, we have(r + s)v = rv + sv.

This are the axioms which are necessary to show vector space .

Note :- A plane vector is an ordered pair (a1, a2) of real numbers and A space vector is an ordered triples (a1, a2, a3) of real numbers.

General properties of vector space:-

The following properties are consequences of the vector spaceaxioms.

- The zero vector is unique.
- 0u = 0 for all u ∈ V .
- k0 = 0 for all scalar k.
- The additive inverse of a vector is unique.
- For all u ∈ V , its additive inverse is given by - u = (-1)u.
- If k is a scalar and u ∈ V such that ku = 0 then eitherk = 0 or u =0.
- If au=bu and u be non zero element of V then a=b.
- If au =av and a be non zero scalar then u=v.
- If a € F and u, v €V then a (u-v) = au-av.

Examples of vector spaces:-

1. The set of all vectors in a plane over field of real numbers is a vector space with respect to vector addition and scalar multiplication.

Proof: let V be the vector space and R be the set of real numbers.

The set of all vectors in a plane is of form (x, y), Where x, y € R

I.e V = {(x, y): x, y € R}

- **Internal Binary operations on V:** let u, v € V s .t u=(u1, u2) and

  v= (v1, v2); u1, u2, v1,v2 € V

  Then u+v= (u1,u2)+(v1,v2)= (u1+v1, u2+v2) € V

Thus addition operation is well defined inV , soIt is internal Binary operation on V.

- **External binary operation on V:**let u € V and k €R(field of scalars)

u = (u1,u2); u1, u2 € V

Then ku = k(u1, u2)= (ku1,ku2) € V

Thus scalar multiplication is well defined so It is external binary operation on V.

Now to show the set of plane vectors over field of real is a vector space , we prove all remaining axioms of vector space with respect to addition and scalar multiplication.

- **1(a). Associativity:** For all u, v,w € V s.t u= =(u1, u2) , v= (v1, v2) and w= (w1, w2); u1, u2, v1,v2 , w1,w2 € V
(u+v)+w=((u1,u2)+(v1,v2))+(w1,w2)=(u1+v1+w1,u2+v2+w2)
Similarly, u+ (v+w)= (u1+v1+w1, u2+v2+w2)

Thus, associativity property satisfied.

- **1(b). Commutativity:**For all u, v € V s.t u= =(u1, u2) and v= (v1, v2) ; u1, u2, v1,v2 € V

u + v= (u1,u2)+(v1,v2)= (u1+v1, u2+v2)= (u, v)

Thus , commutativity property satisfied.

- **1(c) Existence of additive Identity:** For every vector, there exist a zero vector (0 € V) s.t u+0= u=0+u; where u€ V
Thus, Identity property satisfied.
- **1(d) Existence of additive inverse:** For every vector u €V, there exist a vector -u € V such that u+(-u)= 0= (-u) +u.
Thus , inverse property satisfied.
- **1(e) Scalar multiplication is distributive over addition in V:** For u , v € V and k€ R such that

k (u+v)= k ((u1,u2)+(v1,v2))= (ku1, ku2)+(kv1, kv2)

=(ku1+ kv1, ku2+ kv2)=k(u1, u2)+ k(v1, v2)

= ku+kv

Thus , k (u+v)= ku+kv

❖ **1(f) Scalar multiplication is distributive over addition in F:** For every u € V (u1, u2€ V) and k1 , k2 € R such that

u (k1+k2) = (u1, u2)(k1+k2) = (k1u1 + k2u1)+(k1u2+ k2u2)

= k1(u1, u2)+ k2 (u1, u2)

= k1 u+ k2 u

Thus,. u (k1+k2)= k1 u+ k2 u

❖ **1(g) Unit property :** For u€V and 1 be the multiplicative Identity of R

Then, 1. u = 1. (u1, u2) = (u1, u2) = u ; u1, u2€V

Thus, unit property satisfied.

❖ **1(h)** For k1, k2 € R and u€V

Then, (k1k2)u= (k1k2)(u1,u2)= (k1k2u1, k1k2u2)= k1(k2u1,k2u2)

= k1(k2(u1, u2))= k1(k2u)

Thus, (k1k2)u = k1(k2u)

Thus, all properties of vector space are satisfied , so the set of all vector space in a plane over the field of real numbers form a vector space over R.

2.The set of all matrices of type mx n where m, n are fixed positive integers is a vector space over R with respect to matrix addition and matrix multiplication of matrix by a scalar.

3.The set M of order 2x2 real matrices and vector addition and scalar multiplication is defined as given below:

A+ B= A.B and k.A = kA for all A, B € M(2x2) and k€ R.

Proof: This example is not a vector space because It does not satisfy one property of vector space we prove this by taking an example :

Let A =(1 2 3 4) and B= (2 3 4 2) € M (2x2) i.e A and B be two matrices of M (2x2).

Now , A+B = A.B= (1 2 3 4) + (2 3 4 2)= (1 2 3 4)* (2 3 4 2)

= (10 7 22 17)

Also , B+A= B. A = (2 3 4 2) +(1 2 3 4)= (2 3 4 2)* (1 2 3 4)

= (11 16 10 16)

Thus, A+B is not equal to B+A i.e commutativity property is not satisfied.

Hence this is not a vector space over R.

4.The set of all polynomials over field F with constant term zero form a vector space w.r.t vector addition and scalar multiplication.

5. The set of all real valued continuous function on interval [0,1] forms a vector space over the field of real numbers.

6. The set of all polynomials over R with constant term 1 does not form a vector space .

Proof: Because if we take a polynomial whose constant term is one .

Then , the property closed under addition is not satisfied. Hence It is

not a vector space.

7. The set of all polynomials with positive coefficients does not form a Vector space because it is not closed under scalar multiplication.

8. Any Field K(R, Z, Q, I,C) forms a vector space over itself.

Proof: We consider any field which shows that field forms a vector space over itself.

Let F be any field . We know that if F is a field then it has two binary operations say addition and multiplication .

1. F Field forms an abelian group with respect to addition.

2. We satisfied remaining properties of scalar multiplication such as:

- a (u+v)= au+ av, for all a, u,v €F
- (a+b)u= au+ av, for all a, b, u € F
- a (bu)= (ab) u , for all a, b, u € F
- 1.u = u, for all u €F

ISSN 2321 –0567
Vol. VIII, Issue II (Oct.- 2020)

Hence Field F is a vector space over itself

Note:- C is a vector space over C, R , Z, Q because C is a Field and R, Q,Z be the subfield. Similarly R be a vector space over R , Q, Z and Z be a vector space over Z.

Definition of Subspace:

Suppose W be a subset of a vector space V. Then W be a subspace of V if following properties are holds:

- Zero vector belongs to set W i.e 0 € W.
- For every u ,v € W , W is closed under vector addition i.e u+ v € W.
- For every u, v € W, k€ F, W is closed under scalar multiplication i.e k. u € W.

In other words, Subspace may also be defined if we combine both properties which is equivalent to single property that is:

- For every u, v € W and a, b € F, the linear combination a.u + b.v € W.

Note: The vector space V and Zero space {0} are called **trivial or improper** subspaces of V. All other subspaces , if any, are called **proper** subspaces .

Some important theorems of vector subspace:

1. **Every non- empty subset W of a vector space V is a subspace of V iff W is closed under vector addition and scalar multiplication.**

Proof: We prove this theorem by giving examplewhich satisfied these two properties of vector subspace:

Let W = { (x, y, z) : x,y,z € R} , where R be the field of real numbers

Let u= (x1, y1, z1) and v= (x2, y2 , z2) , x1, x2, y1, y2, z1,z2 € R

1).u + v = (x1, y1, z1) + (x2, y2 , z2) = (x2+ x2, y1+y2, z1+z2)

= (x' , y' , z') € W : x' , y' , z' € R

2). a. u = a. (x, y, z) = (ax, ay, az) € W ; a € R and x, y, z € W

Both properties of vector subspace are satisfied, so W is a vector subspace of V over R.

2.Every non- empty subset W of a vector space V is a subspace of V iff a.u+v

€ W , for all u, v € W and a € F.

3.Every non- - empty subset W of a vector space V is a subspace of vector

Space iff two conditions are satisfied:

- **For all u, v € W , a € F , u - v € W and a. u € W.**

4.Everynon- empty subset W of a vector space V is a subspace of V iff a.u + b.v € W , a, b € F and u, v € W.

5 The intersection of arbitrary family of subspaces of s vector space V over F is a subspace of V, but the union of two subspaces of a vector space V may not be a subspace of V over F.

We consider a example to show that union of two subspaces of a vector space V over F may not be a subspave of V over F

Let W1= {(0,0,z): z € R} and W2= {(x, 0,0): x € R}

W1 and W2 be the two subspaces of vector space V over the set of real numbers R.

Let u, v € W1UW2, so that u € W1 and v € W2

⇨ u = (0,0,z) and v= (x, 0,0) , for all u, v € W and a € F. where x, z € R

So W1UW2= { w: w= (0,0,z) or w= (x, 0,0) , for all x, z € R}

Now , u+ v = (0,0,z)+(x, 0,0)= (x, 0,z)

We know that (x, 0,z) does not belong to W1 and W2 both

Therefore , u+v does not belong to W1UW2.

⇨ W1UW2 is not closed under vector addition

Hence, union of two subspaces need not be subspace of V.

6.The union of two subspaces is a subspace iff one is contained in another.

Geometrical Interpretation of subspace of R^3:-

- A line in the subspace R^3 is a subspace iff it passes through the origin.

I.e {(x ,y,z) € R^3: ax + by + cz=0, dx + ey + fz=0} is the mathematical interpretation of this line.

- A plane in the space R^3 is a subspace iff it passes through the origin.

I.e {(x ,y,z) € R^3: ax + by + cz=0 } is the mathematical interpretation of this plane.

Examples of subspaces:-

1. **Let V be a vector space given by V= IR^3 = { (x ,y,z): x, y,z € IR^3} and W= { (x ,y,z) € IR^3: ax + by + cz= 0} be a subspace of vector space V over F.**
2. **Let V= Mn(¢) be the vector space of all n- square matrices over complex numbers, then**
   - **The set of all diagonal matrices**
   - **The set of all scalar matrices**
   - **The set of all lower and upper triangular matrices**
   - **The set of all super lower and super upper triangular matrices**
   - **The set of all backward diagonal and backward scalar matrices**
   - **The set of all symmetric and skew- symmetric matrices**

toform subspaces of vector space V .

3. **Let V= Mn(¢) over set of real number IR and V = Mn (IR) over IR , then including all above matrices of example 2 , the set of all Hermitian and Non Hermitian matrices also form a subspace of a vector space V over IR.**
4. **Let V (R) be the vector space of all functions from IR to IR .then the set of all even functions, set of all odd functions and set of all continuous functions are subspace of a vector space V over IR.**
5. **Linear sum of subspaces:-**

Definition:-Let W1 and W2 be two subspaces of a vector space V over F then the linear sum of two subspaces be defined as:

W1+ W2= {w1 +w2 ; w1 € W1 and w2 € W2}

Direct sum of subspaces:-Definition:-

Let W1 and W2 be two subspaces of a vector space V over F ,then a vector space V is said to be direct sum of two subspaces if every vector of V can be uniquely expressed as :**v = w1+ w2 ; v € V, w1 € W1 and w2 € W2**

In other words , Direct sum may also be defined such that a vector space V over

F is said to be direct sum of its subspace W1 and W2 are that

- V = W1+W2
- Intersection of W1 and W2 is equal to zero.

Examples of linear sum and Direct sum:-

1. Consider a vector space V3(IR) and let W1 and W2 be two subspaces of vector space V given by:

W1= {(a, 0,0): a € IR}and W2 = {(0, b, c): b, c € IR}

Then (a, b,c)€ V can be written as sum of W1 and W2 as:

(a, b,c)= (a,0,0)+ (0,b,c) , where (a, 0,0) € W1 , (0,b,c) € W2

Thus (a,b,c) can be expressed only one way in sum of subspacesW1and W2,soV be the direct sum of W1 and W2 subspaces.

2. Consider a vector space V3(IR) and let W1 and W2 be two subspaces of a vector space V given by:

W1= {(a, b,0): a, b € IR} and W2 = { (a, b,c): a,b,c € IR}

Then (a,b,c)€ V can be written as a sum of W1 and W2 as:

(a,b,c) = (a/2, b/2, 0) + (a/2, b/2,c), where (a/2 ,b/2,0) € W1 and (a/2, b/2,c) € W2

(a,b,c) = (3a/4, 3b/4,0) + (a/4, b/4,c), where (3a/4 ,3b/4,0) € W1 and (a/4, ba4,c) € W
Thus (a,b,c) can be written as more than one ways i.e not uniquely expressed as sum of subspaces of W1 and W2.

Hence this is a example of linear sum of subspaces not of direct sum of subspaces.

Conclusion:-

At last, we can say that this paper reviews the uses of linear algebra concept in various areas. In linear algebra , the concept of vector space and it's subspaces and the use of vector space is in quantum mechanics.

Vector space are natural objects to study if you want to learn about linear algebra. If once you have a grasp of this way of viewing linear algebra, then you will have a much easier time understanding a concept like Fourier transform . Many such topic of mathematics are interrelated with linear algebra and clearly understand by the use of vector space .

References:-

1. Lipschutz, S. and Lipson, M.L. (2009) Schuam Outline Series Linear Algebra. 4th Edition, Mcgraw Hill Publishers, New York, 116-170.
2. Hoffman, K. And R. Kunze(1997), Eastern Economy Edition Linear Algebra. Second Edition, Asoke K. Ghosh, New Delhi, 28-58.
3. Robert Stoll, R. (2012) linear algebra and matrix theory,Second Edition, Dover publication Inc., New York, 42.
4. Roman, S.(2005) Advanced linear algebra , Second Edition, Springer publication , California, 33-39.

5. Schneider Hans and G. Phillip Barker (1973) Matrices and linear algebra,Second Edition,Dover publication, New York, 89- 86.

6. Blyth T.S and E.F . Robertson (2005) Basic Linear Algebra , Second Edition , Springer – Verlag London limited , 69-75.

7. Kwak J.H and Sungpyo Hung(2004), Linear Algebra , Second Edition ,Springer published by Birkhauser Boston, New York, 75-79.

8. https://math.vanderkilt.edu.